Time-Inconsistency: Problems and Mathematical Theory

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Outline

1. **Introduction: Time-Consistency**
2. **Time-Inconsistent Problems**
3. **Equilibrium Strategies**
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1. Introduction: Time-Consistency

Continuous Compound Interest

— Exponential Discounting.

$P(0)$ — initial principal

r — annual interest rate

$P(t) = P(0)e^{rt}$ — Amount at the end of t-th year (compounded continuously)

For any given future times $T > t > 0$, from

$$P(T) = P(0)e^{rT}, \qquad P(t) = P(0)e^{rt},$$

one has

$$P(T) = P(t)e^{r(T-t)}, \qquad 0 < t < T,$$

or, equivalently,

$$P(t) = P(T)e^{-r(T-t)}, \qquad 0 < t < T.$$

This is the value (price) at t of a payoff $P(T)$ at T.

$e^{-r(T-t)}$ — **exponential discounting**.

For any $0 < t_1 < t_2 < T$, one has

$$e^{r(T-t_1)}P(t_1) = P(T) = e^{r(T-t_2)}P(t_2).$$

Therefore,

$$\boxed{\text{possess } P(t_1) \text{ at } t_1} \asymp \boxed{\text{possess } P(t_2) \text{ at } t_2}$$

This is called the **Time-Consistency** of **exponential discounting**

Preferred Choice: Assume that annual rate is $r = 10\%$

Option (A): Get \$100 today (December 28, 2015).

Option (B): Get \$105 ($> 100(1 + \frac{r}{12})$) on January 28, 2016.

Option (A$'$): Get \$110 ($= 100 \times 1.10$) on December 28, 2016.

Option (B$'$): Get \$115.50 ($> 110(1 + \frac{r}{12})$) on January 28, 2017.

For a **time-consistent** person,

$$
\begin{aligned}
&(\mathrm{A}) \asymp (\mathrm{A}'), \qquad (\mathrm{B}) \asymp (\mathrm{B}'), \\
&(\mathrm{B}) \succ (\mathrm{A}), \qquad (\mathrm{B}') \succ (\mathrm{A}').
\end{aligned}
$$

(We will come back to this example later)

Semigroups: Consider

$$\begin{cases} \dot{X}(s) = b(s, X(s)), & s \in [t, T], \\ X(t) = x. \end{cases}$$

Suppose for any $(t, x) \in [0, T) \times \mathbb{R}^n$, the above admits a unique solution $X(\cdot\,; t, x)$. Then for any $\tau \in (t, T)$,

$$X(s; \tau, X(\tau; t, x)) = X(s; t, x), \qquad s \in [\tau, T].$$

The restriction $X(\cdot\,; t, x)\big|_{[\tau, T]}$ is the solution of the equation starting from $(\tau, X(\tau; t, x))$. **A (nonlinear) semigroup property.**

Dynamic Programming/Feymann-Kac formula: Consider

$$\dot{X}(s) = b(s, X(s)), \qquad s \in [0, T],$$

$$J(t, X(t)) = h(X(T)) + \int_t^T g(s, X(s))ds.$$

For $\tau \in (t, T)$,

$$J(t, X(t)) = h(X(T; t, X(t))) + \int_t^T g(s, X(s; t, X(t)))ds$$

$$= h(X(T; \tau, X(\tau; t, X(t)))) + \int_\tau^T g(s, X(s; \tau, X(\tau; t, X(t))))ds$$

$$+ \int_t^\tau g(s, X(s; t, X(t)))ds = J(\tau, X(\tau)) + \int_t^\tau g(s, X(s))ds.$$

Extended semigroup property. (Special case: $h(x) = x, g = 0$)

This leads to

$$\begin{cases} J_t(t,x) + J_x(t,x)b(t,x) + g(t,x) = 0, \\ J(T,x) = h(x). \end{cases} \tag{1}$$

Linear **Hamilton-Jacobi** type equation.

Another viewpoint: The solution $J(t,x)$ of PDE (1) admits representation:

$$J(t,x) = h(X(T;t,x)) + \int_t^T g(s, X(s;t,x))ds.$$

This is a deterministic **Feynman-Kac formula**.

Optimal Control Problem: Consider

$$\begin{cases} \dot{X}(s) = b(s, X(s), u(s)), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with (scalar) cost functional

$$J(t, x; u(\cdot)) = h(X(T)) + \int_t^T g(s, X(s), u(s))ds,$$

where

$$\mathcal{U}[t, T] = \{ u : [t, T] \to U \mid u(\cdot) \text{ is measurable } \}.$$

Problem (C). For given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)) \equiv V(t, x).$$

Bellman Optimality Principle: For any $\tau \in [t, T]$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \Big[\int_t^\tau g(s, X(s), u(s))ds + V\big(\tau, X(\tau; t, x, u(\cdot))\big) \Big].$$

Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be optimal for $(t,x) \in [0, T) \times \mathbb{R}^n$.

$$\begin{aligned} V(t,x) &= J(t,x;\bar{u}(\cdot)) = \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds \\ &\quad + J\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot)); \bar{u}(\cdot)\big|_{[\tau, T]}\big) \\ &\geq \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds + V\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot))\big) \\ &\geq \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \int_t^\tau g(s, X(s), u(s))ds \\ &\quad + V\big(\tau, X(\tau; t, x, u(\cdot))\big) = V(t,x). \end{aligned}$$

Thus, all the equalities hold.

Consequently,

$$
\begin{aligned}
J\big(\tau, \bar{X}(\tau); \bar{u}(\cdot)\big|_{[\tau,T]}\big) &= V(\tau, \bar{X}(\tau)) \\
&= \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J\big(\tau, \bar{X}(\tau); u(\cdot)\big), \quad \text{a.s.}
\end{aligned}
$$

Hence, $\bar{u}(\cdot)\big|_{[\tau,T]} \in \mathcal{U}[\tau, T]$ is **optimal** for $\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot))\big)$.

This is called the **time-consistency** of Problem (C).

Definition. A problem involving a decision-making is said to be **time-consistent** if

an **optimal** decision made at a given time t
will remain **optimal** at any time $s > t$.

If the above is not the case, the problem is said to be **time-inconsistent**.

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If the problem under consideration is time-consistent, then once an optimal decision is made, we will not regret afterwards!

If the whole world is **time-consistent**,

then the things are too **ideal**, the life will be much **easier**!

But, it might also be a little or too **boring**

(exciting to have some challenges)!

Fortunately (unfortunately?), the life is not that ideal!

(Challenges are around!)

Time-inconsistent problems exist almost everywhere!

2. Time-Inconsistent Problems

In reality, problems are **hardly time-consistent**:

An optimal decision/policy made at time t, more than often, will not stay optimal, thereafter.

Main reason: When building the model, describing the utility/cost, etc., the following are used:

subjective Time-Preferences and

subjective Risk-Preferences.

- **Time-Preferences:**

Most people do not discount exponentially! Instead, they over discount on the utility of immediate future outcomes.

* Overreaction without thinking the consequences (bad temper and impatience lead to unnecessary fighting,...)

* Break promise, delay planned projects (fail to meet deadlines, such as refereeing papers, quit smoking, ...)

* Shopping using credit cards (meeting immediate satisfaction, big discount, buy one get one free,...)

* Unintentionally pollute the environment due to over-development

* Corruption, without thinking consequences

....

Doing things not because you need to do
but because you like to do.

Not Doing things not because you do not need to do
but because you do not like to do.

* D. Hume (1739), “A Treatise of Human Nature”

“**Reason is, and ought only to be the slave of the passions.**”

More than often, people doing things is due to their **passions**.

* A. Smith (1759), "The Theory of Moral Sentiments"

Utility is not intertemporally sparable but rather that past and future experiences, jointly with current ones, provide current utility.

Roughly, in mathematical terms, one should have

$$U(t, X(t)) = f(U(t-r, X(t-r)), U(t+\tau, X(t+\tau)),$$

where $U(t, X)$ is the utility at (t, X).

Exponential discounting: $\lambda_e(t) = e^{-rt}, \quad r > 0$ — discount rate

Hyperbolic discounting: $\lambda_h(t) = \frac{1}{1+kt}$ — a hyperbola

If let $k = e^r - 1$, i.e., $e^{-r} = \lambda_e(1) = \lambda_h(1) = \frac{1}{1+k}$, then

$$\lambda_e(t) = e^{-rt} = \frac{1}{(1+k)^t}, \qquad \lambda_h(t) = \frac{1}{1+kt}.$$

For $t \sim 0$, $t \mapsto \frac{1}{1+kt}$ decreases faster than $t \mapsto \frac{1}{(1+k)^t}$:

$$\lambda_h'(0) = -k < -\ln(1+k) = \lambda_e'(0),$$

Hyperbolic discounting actually appears in **people's behavior**.

Come back to a previous example: Annual rate is 10%

Option (A): Get \$100 today (December 28, 2015).

Option (B): Get \$105 ($> 100(1+\frac{r}{12})$) on January 28, 2016.

Option (A$'$): Get \$110 ($= 100 \times 1.10$) on December 28, 2016.

Option (B$'$): Get \$115.50 ($> 110(1+\frac{r}{12})$) on January 28, 2017.

For a **time-consistent** person,

$$\text{(A)} \asymp \text{(A}'\text{)}, \qquad \text{(B)} \asymp \text{(B}'\text{)},$$
$$\text{(B)} \succ \text{(A)}, \qquad \text{(B}'\text{)} \succ \text{(A}'\text{)}.$$

However, for an **uncertainty-averse** person,

$$\text{(A)} \succ \text{(B)}, \qquad \text{(B}'\text{)} \succ \text{(A}'\text{)}.$$

Magnifying the example:

Option (A): Get \$1M today (December 28, 2015).

Option (B): Get \$1.05M ($> 1M(1 + \frac{r}{12})$) on January 28, 2016.

Option (A′): Get \$1.1M ($= 1M \times 1.10$) on December 28, 2016.

Option (B′): Get \$1.155M($> 1.1M(1 + \frac{r}{12})$) on January 28, 2017.

For an **uncertainty-averse** person,

$$\text{(A)} \succ \text{(B)}, \qquad \text{(B}'\text{)} \succ \text{(A}'\text{)}.$$

The feeling is stronger?

* Palacious–Huerta (2003), survey on history
* Strotz (1956), Pollak (1968), Laibson (1997), ...
* Finn E. Kydland and Edward C. Prescott, (1977)
  (2004 Nobel Prize winners)
  (classical optimal control theory not working)
* Ekeland–Lazrak (2008), Yong (2011, 2012)

- **Risk-Preferences:**

Consider two investments whose returns are: R_1 and R_2 with

$$\mathbb{P}(R_1 = 100) = \frac{1}{2}, \qquad \mathbb{P}(R_1 = -50) = \frac{1}{2},$$
$$\mathbb{P}(R_2 = 150) = \frac{1}{3}, \qquad \mathbb{P}(R_2 = -60) = \frac{2}{3}.$$

Which one you prefer?

$$\mathbb{E}R_1 = \frac{1}{2}100 + \frac{1}{2}(-50) = 25,$$
$$\mathbb{E}R_2 = \frac{1}{3}150 + \frac{2}{3}(-60) = 10.$$

So R_1 seems to be better.

* St. Petersburg Paradox: (posed by Nicolas Bernoulli in 1713)

$$\mathbb{P}(X = 2^n) = \frac{1}{2^n}, \qquad n \geq 1,$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} 2^n \mathbb{P}(X = 2^n) = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \infty.$$

Question: How much are you willing to pay to play the game?

How about \$10,000? Or \$1,000? Or ???

In 1738, Daniel Bernoulli (a cousin of Nicolas) introduced **expected utility**: $\mathbb{E}[u(X)]$. With $u(x) = \sqrt{x}$, one has

$$\mathbb{E}\sqrt{X} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = 1 + \sqrt{2}.$$

* 1944, von Neumann–Morgenstern: Introduced "rationality" axioms: Completeness, Transitivity, Independence, Continuity.

Standard stochastic optimal control theory is based on the expected utility theory.

- Decision-making based on expected utility theory is **time-consistent**.
- In classical expected utility theory, the probability is **objective**.
- It is controversial whether a probability should be **objective**.
- Early relevant works: Ramsey (1926), de Finetti (1937)

Allais Paradox (1953). $\Omega = \{1, 2, \cdots, 100\}$, $\mathbb{P}(\omega) = \frac{1}{100}, \forall \omega \in \Omega$.

$$X_1(\omega) = 100\chi_{\{1 \le \omega \le 100\}}, \qquad X_2(\omega) = 200\chi_{\{1 \le \omega \le 70\}},$$
$$X_3(\omega) = 100\chi_{\{1 \le \omega \le 15\}}, \qquad X_4(\omega) = 200\chi_{\{1 \le \omega \le 10\}}.$$

Most people have the following preferences:

$$X_2 \prec X_1, \qquad X_3 \prec X_4.$$

If there exists a utility function $u : \mathbb{R} \to \mathbb{R}^+$ such that

$$X \prec Y \quad \Longleftrightarrow \quad \mathbb{E}[u(X)] < \mathbb{E}[u(Y)],$$

then

$$X_2 \prec X_1 \quad \Rightarrow \quad \mathbb{E}[u(X_2)] = 0.7u(200) < u(100) = \mathbb{E}[u(X_1)],$$
$$X_3 \prec X_4 \quad \Rightarrow \quad \mathbb{E}[u(X_3)] = 0.15u(100) < 0.1u(200) = \mathbb{E}[u(X_4)],$$

Thus, $1.05u(100) < 0.7u(200) < u(100)$, a contradiction.

Ellesberg's Paradox (1961). In an urn, there are 90 balls,

	30	60	
	Red	Black	White
X_R	\$100	0	0
X_B	0	\$100	0
$X_{R\cup W}$	\$100	0	\$100
$X_{B\cup W}$	0	\$100	\$100

Most people have the following preferences: (ambiguity-averse)

$$X_B \prec X_R, \qquad X_{R\cup W} \prec X_{B\cup W}.$$

$$\mathbb{P}(R) = \frac{1}{3}, \quad \mathbb{P}(B) \in [0, \frac{2}{3}], \quad \mathbb{P}(B\cup W) = \frac{2}{3}, \quad \mathbb{P}(R\cup W) \in [\frac{1}{3}, 1].$$

$$\mathbb{P}(B \cup W) = \mathbb{P}(B) + \mathbb{P}(W), \quad \mathbb{P}(R \cup W) = \mathbb{P}(R) + \mathbb{P}(W).$$

$$X_B \prec X_R, \qquad X_{R\cup W} \prec X_{B\cup W}.$$

$$\mathbb{P}(R) = \frac{1}{3}, \quad \mathbb{P}(B) \in [0, \frac{2}{3}], \quad \mathbb{P}(B \cup W) = \frac{2}{3}, \quad \mathbb{P}(R \cup W) \in [\frac{1}{3}, 1].$$

$$\mathbb{P}(B \cup W) = \mathbb{P}(B) + \mathbb{P}(W), \quad \mathbb{P}(R \cup W) = \mathbb{P}(R) + \mathbb{P}(W).$$

If there exists a utility function $u : \mathbb{R} \to \mathbb{R}^+$ such that

$$X \prec Y \quad \Longleftrightarrow \quad \mathbb{E}[u(X)] < \mathbb{E}[u(Y)],$$

then

$$\begin{aligned}
X_{R\cup W} \prec X_{B\cup W} \quad &\Longleftrightarrow \quad u(100)\mathbb{P}(R \cup W) < u(100)\mathbb{P}(B \cup W) \\
\Longleftrightarrow \quad & u(100)\mathbb{P}(R) = u(100)[\mathbb{P}(R \cup W) - \mathbb{P}(W)] \\
& \qquad < u(100)[\mathbb{P}(B \cup W) - \mathbb{P}(W)] = u(100)\mathbb{P}(B) \\
\Longleftrightarrow \quad & X_R \prec X_B,
\end{aligned}$$

a contradiction.

Relevant Literature:

* Subjective expected utility theory (Savage 1954)

* Mean-variance preference (Markowitz 1952)
  leading to nonlinear appearance of conditional expectation

* Choquet integral (1953)
  leading to Choquet expected utility theory

* Prospect Theory (Kahneman–Tversky 1979)
  (Kahneman won 2002 Nobel Prize)

* Distorted probability (Wang–Young–Panjer 1997)
  widely used in insurance/actuarial science

* BSDEs, g-expectation (Peng 1997)
  leading to time-consistent nonlinear expectation

* BSVIEs (Yong 2006,2008)
  leading to time-inconsistent dynamic risk measure

Recent Relevant Literatures:

* Björk–Murgoci (2008), Björk–Murgoci–Zhou (2013)

* Hu–Jin–Zhou (2012, 2015)

* Yong (2012, 2013, 2014, 2015)

- **A Summary:**

Time-Preferences: (Exponential/General) Discounting.

Risk-Preferences: (Subjective/Objective) Expected Utility.

Exponential discounting + **objective** expected utility/disutility leads to **time-consistency**.

Otherwise, the problem will be **time-inconsistent**.

Time-consistent solution:

Instead of finding an optimal solution
(which is **time-inconsistent**),

find an equilibrium strategy
(which is **time-consistent**).

Sacrifice some immediate satisfaction,

save some for the future

(retirement plan, controlling economy growth speed, ...)

3. Equilibrium Strategies

A General Formulation:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), & s \in [t,T], \\ X(t) = x, \end{cases}$$

with

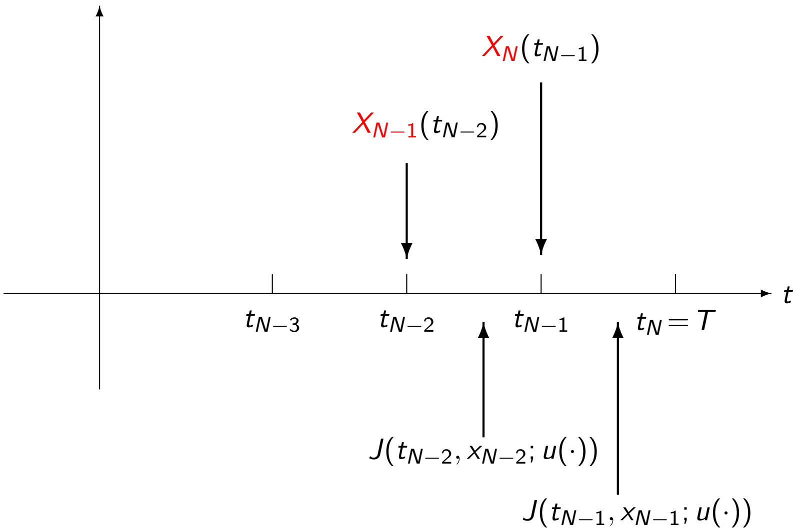
$$J(t, x; u(\cdot)) = \mathbb{E}_t\Big[\int_t^T g(t, s, X(s), u(s))ds + h(t, X(T))\Big].$$

$$\mathcal{U}[t, T] = \Big\{ u : [t, T] \to U \mid u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \Big\}.$$

Problem (N). For given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)).$$

This problem is **time-inconsistent**.


$X_N(t_{N-1})$
$X_{N-1}(t_{N-2})$
t
t_{N-3}
t_{N-2}
t_{N-1}
$t_N = T$
$J(t_{N-2}, x_{N-2}; u(\cdot))$
$J(t_{N-1}, x_{N-1}; u(\cdot))$

Idea of Seeking Equilibrium Strategies.

• Partition the interval $[0, T]$:

$$[0, T] = \bigcup_{k=1}^{N} [t_{k-1}, t_k], \qquad \Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N.$$

• Solve an optimal control problem on $[t_{N-1}, t_N]$, with cost functional:

$$J_N(u) = \mathbb{E}\Big[h(t_{N-1}, X(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-1}, s, X(s), u(s))ds\Big],$$

obtaining optimal pair $(X_N(\cdot), u_N(\cdot))$, depending on the initial pair (t_{N-1}, x_{N-1}).

• Solve an optimal control problem on $[t_{N-2}, t_{N-1}]$ with a **sophisticated** cost functional:

$$J_{N-1}(u) = \mathbb{E}\Big[h(t_{N-2}, X(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-2}, s, X_N(s), u_N(s))ds + \int_{t_{N-2}}^{t_{N-1}} g(t_{N-2}, s, X(s), u(s))ds\Big].$$

• By induction to get an approximate equilibrium strategy, depending on Π.

• Let $\|\Pi\| \to 0$ to get a limit.

Definition. $\Psi : [0,T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* if for any $x \in \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0,T], \\ \bar{X}(0) = x \end{cases}$$

admits a unique solution $\bar{X}(\cdot)$. For some $\Psi^\Pi : [0,T] \times \mathbb{R}^n \to U$,

$$\lim_{\|\Pi\| \to 0} d\Big(\Psi^\Pi(t,x), \Psi(t,x)\Big) = 0,$$

uniformly for (t,x) in any compact sets, where $\Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, and

$$\begin{aligned} & J^k\big(t_{k-1}, X^\Pi(t_{k-1}); \Psi^\Pi(\cdot)\big|_{[t_{k-1},T]}\big) \\ & \leq J^k\big(t_{k-1}, X^\Pi(t_{k-1}); u^k(\cdot) \oplus \Psi^\Pi(\cdot)\big|_{[t_k,T]}\big), \quad \forall u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k], \end{aligned}$$

$J^k(\cdot)$ — sophisticated cost functional.

$$
\begin{cases}
dX^{\Pi}(s) = b(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))ds \\
\qquad\qquad +\sigma(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))dW(s), \quad s \in [0, T], \\
X^{\Pi}(0) = x
\end{cases}
$$

$$
[u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k, T]}](s) = \begin{cases}
u^k(s), & s \in [t_{k-1}, t_k), \\
\Psi^{\Pi}(s, X^k(s)), & s \in [t_k, T],
\end{cases}
$$

$$
\begin{cases}
dX^k(s) = b(s, X^k(s), u^k(s))ds \\
\qquad\qquad +\sigma(s, X^k(s), u^k(s))dW(s), \quad s \in [t_{k-1}, t_k), \\
dX^k(s) = b(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))ds \\
\qquad\qquad +\sigma(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))dW(s), \quad s \in [t_k, T], \\
X^k(t_{k-1}) = X^{\Pi}(t_{k-1}).
\end{cases}
$$

Equilibrium control:

$$\bar{u}(s) = \Psi(s, \bar{X}(s)), \qquad s \in [0, T].$$

Equilibrium state process $\bar{X}(\cdot)$, satisfying:

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0, T], \\ \bar{X}(0) = x \end{cases}$$

Equilibrium value function:

$$V(t, \bar{X}(t)) = J(t, \bar{X}(t); \bar{u}(\cdot)).$$

The previous explained idea will help us to get such a $\Psi(\cdot\,,\cdot)$.

Let $D[0,T] = \{(\tau,t) \mid 0 \le \tau \le t \le T\}$. Define

$$a(t,x,u) = \frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T, \quad \forall (t,x,u) \in [0,T] \times \mathbb{R}^n \times U,$$

$$\mathbb{H}(\tau,t,x,u,p,P) = \operatorname{tr}\big[a(t,x,u)P\big] + \langle\, b(t,x,u), p\,\rangle + g(\tau,t,x,u),$$
$$\forall (\tau,t,x,u,p,P) \in D[0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{S}^n,$$

Let $\psi : \mathcal{D}(\psi) \subseteq D[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n \to U$ such that

$$\mathbb{H}(\tau,t,x,\psi(\tau,t,x,p,P),p,P) = \inf_{u \in U} \mathbb{H}(\tau,t,x,u,p,P) > -\infty,$$
$$\forall (\tau,t,x,p,P) \in \mathcal{D}(\psi).$$

In **classical** case, it just needs

$$H(t,x,p,P) = \inf_{u \in U} \mathbb{H}(t,x,u,p,P) > -\infty,$$
$$\forall (t,x,p,P) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n.$$

Equilibrium HJB equation:

$$\begin{cases}\Theta_t(\tau,t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\Theta_{xx}(\tau,t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle\\ +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(\tau,t,x))\big)=0,\ \ (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}$$

Classical HJB Equation:

$$\begin{cases}\Theta_t(t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)\Theta_{xx}(t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big),\Theta_x(t,x)\,\rangle\\ +g\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)=0,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

or

$$\begin{cases}\Theta_t(t,x)+H\big(t,x,\Theta_x(t,x),\Theta_{xx}(t,x)\big)=0,\quad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

Equilibrium value function:

$$V(t,x) = \Theta(t,t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

It satisfies

$$V(t,\bar{X}(t;x)) = J\big(t,\bar{X}(t;x);\Psi(\cdot)\big|_{[t,T]}\big), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Equilibrium strategy:

$$\Psi(t,x) = \psi(t,t,x,V_x(t,x),V_{xx}(t,x)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Theorem. *Under proper conditions, the equilibrium HJB equation admits a unique classical solution $\Theta(\cdot\,,\cdot\,,\cdot)$. Hence, an equilibrium strategy $\Psi(\cdot\,,\cdot)$ exists.*

4. Open Problems

1. The well-posedness of the equilibrium HJB equation for the case $\sigma(t, x, u)$ is **not independent** of u.
2. The case that ψ is **not unique**, has **discontinuity**, etc.
3. The case that $\sigma(t, x, u)$ is **degenerate**, viscosity solution?
4. **Random** coefficient case (non-degenerate/degenerate cases).
5. The case involving **conditional expectation**.
6. **Infinite horizon** problems.

Thank You!